Understanding the Math Behind Perfect Substitutes

Consider the following utility function:

$$U(x_1, x_2) = ax_1 + bx_2 \tag{1}$$

If a consumer is willing to replace the amount consumed of one good for another at a constant rate, we can represent their preferences through a linear function as shown above.
Faced with the following budget constraint:

$$p_1x_1 + p_2x_2 = m \tag{2}$$

Solve the consumer's maximization problem using the Lagrange method.

Solution

At first, one would try to set up the Lagrangian and maximize as if it were a Cobb-Douglas utility function; the problem is that if it were the case that given the prices and coefficients a and b, the consumer prefers x_1 to x_2, they could (given how the problem is posed) choose negative quantities of x_2 such that this "generates" disposable income to consume more of x_1. Since negative quantities do not make sense in this problem, we must pose two more constraints:

$$x_1 \geq 0 \tag{3}$$

$$x_2 \geq 0 \tag{4}$$

These last 2 constraints can be expressed in the following way (for convenience):

$$-x_1 \leq 0 \tag{5}$$

$$-x_2 \leq 0 \tag{6}$$

Now it is possible to set up the Lagrangian. We must maximize 1.1 subject to 1.2, 1.5, and 1.6

$$\mathcal{L} = ax_1 + bx_2 + \lambda(m - p_1x_1 - p_2x_2) + \mu_1(x_1) + \mu_2(x_2) \tag{7}$$

It is possible to show that if $m > 0$ and $\frac{a}{p_1} \neq \frac{b}{p_2}$, the consumer will choose only positive quantities of one good, which is expected since the goods are perfect substitutes, meaning that at a constant rate, we can replace the consumption of one for another. The conditions stated above mean that the consumer has money and that given the prices, they prefer one good over the other.
The result will be that the consumer consumes only x_1 if $\frac{a}{p_1} > \frac{b}{p_2}$ and consumes only x_2 if $\frac{a}{p_1} < \frac{b}{p_2}$.

Resolution

The first-order conditions are as follows:

$$\frac{\partial L}{\partial x_1} = a - \lambda p_1 + \mu_1 = 0 \tag{8}$$

$$\frac{\partial L}{\partial x_2} = b - \lambda p_2 + \mu 2 = 0 \tag{9}$$

$$p1x1 + p2x2 = m \tag{10}$$

Additionally, since we are dealing with a problem with constraints that can be met with both equality and inequality, we set the complementary slackness conditions:

$$x_1\mu_1 = 0 \tag{11}$$

$$x_2\mu_2 = 0 \tag{12}$$

These two conditions indicate that if some constraint is met with equality (x_1 or x_2 equal to 0), then the value of μ does not concern us (there is no need for it to have a specific value). On the other hand, if they are not met with equality, then it must be true that the constraint does not limit the maximization in any way, so both or some μ will be 0 (indicating that the Lagrangian does not take that constraint into account). We see that from (8) and from (9) we get the following:

$$\frac{a+\mu_1}{p_1} = \lambda \tag{13}$$

$$\frac{b+\mu_2}{p_2} = \lambda \tag{14}$$

We equate the λ values:

$$\frac{b+\mu_2}{p_2} = \frac{a+\mu_1}{p_1} \tag{15}$$

We clear p_1:

$$p_1 = \frac{(a+\mu_1)p_2}{b+\mu_2} \tag{16}$$

We insert it in (10):

$$x_1\frac{(a+\mu_1)p_2}{b+\mu_2} + p_2x_2 = m \tag{17}$$

Let's see the possible values that the variables can take: If (11) must be satisfied, then there are two possibilities

$$x_1 = 0 \vee \mu_1 = 0 \tag{18}$$

Additionally, if (12) must be satisfied, then there are two more possibilities:

$$x_2 = 0 \vee \mu_2 = 0 \tag{19}$$

We have $2^2 = 4$ possible combinations

$$x_1 = 0 \wedge x_2 = 0 \tag{20}$$

$$x_1 \neq 0 \wedge x_2 = 0 \tag{21}$$

$$x_1 = 0 \wedge x_2 \neq 0 \tag{22}$$

$$x_1 \neq 0 \wedge x_2 \neq 0 \tag{23}$$

Let's first consider (20). In this case, (10) and (12) are satisfied. However, (17) is only satisfied if $m = 0$. Therefore, we conclude that the individual will not consume anything only if their income is zero. Regarding the case (23), we see that it necessarily implies that $\mu_1 = \mu_2 = 0$. Therefore, (16) becomes

$$p_1 = \frac{ap_2}{b} \tag{24}$$

$$\frac{a}{b} = \frac{p_1}{p_2} \tag{25}$$

This case only occurs if the individual is indifferent between the 2 goods given the prices. Note that (25) indicates that the slopes of the indifference curve and the slope of the budget constraint must be equal. This can only happen in the particular case that the combinations of x_1 and x_2 do not matter, as long as the individual meets (10), they will have the same utility.

Now let's move on to the last two cases, which are the most relevant. Suppose (21) occurs, then it must be true that $\mu_1 = 0$ due to (11). Since $x_2 = 0$, we see that (10) becomes

$$x_1 p_1 + 0 = m \tag{26}$$

$$x_1 = \frac{m}{p_1} \tag{27}$$

Then, we meet all the conditions. $(\frac{m}{p_1}, 0)$ is a candidate for the maximum.
The reasoning is similar for (22), we see that this possibility implies that $\mu_2 = 0$ due to condition (12). Furthermore, due to (25), it must be true that $x_2 = \frac{m}{p_2}$. Just like in the previous case, all conditions are met and we have another candidate for the maximum: $(0, \frac{m}{p_2})$.

What is the maximum then? Discarding the cases in which the individual has no income and in which they are indifferent between the 2 goods (given the prices), we derive the condition for choosing between x_1 and x_2 by comparing the utilities.
In one case, the utility is:

$$U(\frac{m}{p_1, 0}) = a\frac{m}{p_1} \tag{28}$$

And in the other:

$$U(0, \frac{m}{p_2}) = b\frac{m}{p_2} \tag{29}$$

Therefore, the individual will choose to consume x_1 (with all their income) if the utility it provides is greater than the case in which the individual chooses to consume x_2 (with all their income):

$$a\frac{m}{p_1} > b\frac{m}{p_2} \tag{30}$$

$$\frac{a}{p_1} > \frac{b}{p_2} \tag{31}$$

$$\frac{a}{b} > \frac{p_1}{p_2} \tag{32}$$

Conclusion:
The optimal consumer choice in the case of perfect substitutes will depend on the slope of the indifference curves compared to the slope of the budget constraint.